The Trojan Horse Method: an Indirect Technique in Nuclear Astrophysics.

A. M. Mukhamedzhanov,[1] C. Spitaleri,[2] and R. E. Tribble[1]

[1]*Cyclotron Institute, Texas A&M University, College Station, TX 77843*
[2]*DMFCI, Universit di Catania, Catania, Italy and INFN - Laboratori Nazionali del Sud, Catania, Italy*
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The Trojan Horse (TH) method is a powerful indirect technique that provides information to determine astrophysical factors for rearrangement processes at astrophysically relevant energies. A short coming for understanding the reliability of the technique has been determining the importance of nuclear and Coulomb effects on the energy dependence of the yield. Using a simple model, we demonstrate that off-energy-shell and Coulomb effects in the entry channel and the final state nuclear interactions do not change the energy dependence of the astrophysical factor extracted from the TH reaction. Some examples are presented.

The presence of the Coulomb barrier for colliding charged nuclei makes nuclear reaction cross sections at astrophysical energies so small that their direct measurement in the laboratory is very difficult, or even impossible. Consequently indirect techniques often are used to determine these cross sections. The Trojan Horse (TH) method is a powerful indirect technique which allows one to determine the astrophysical factor for rearrangement reactions. The TH method, first suggested by Baur [1], involves obtaining the cross section of the binary $x + A \rightarrow b + B$ process at astrophysical energies by measuring the two-body to three-body $(2 \rightarrow 3)$ process, $a + A \rightarrow y + b + B$, in the quasifree (QF) kinematics regime, where the "Trojan Horse" particle, $a = (x\,y)$, is accelerated at energies above the Coulomb barrier. After penetrating through the Coulomb barrier, nucleus a undergoes breakup leaving particle x to interact with target A while projectile y flies away. From the measured $a + A \rightarrow y + b + B$ cross section, the energy dependence of the binary subprocess, $x + A \rightarrow b + B$, is determined.

The main advantage of the TH method is that the extracted cross section of the binary subprocess does not contain the Coulomb barrier factor. Consequently the TH cross section can be used to determine the energy dependence of the astrophysical factor, $S(E)$, of the binary process, $x + A \rightarrow b + B$, down to zero relative kinetic energy of the particles x and A without distortion due to electron screening [2, 3]. The absolute value of $S(E)$ must be found by normalization to direct measurements at higher energies. At low energies where electron screening becomes important, comparison of the astrophysical factor determined from the TH method to the direct result provides a determination of the screening potential.

Even though the TH method has been applied successfully to many direct and resonant processes (see [4] and references therein), there are still reservations about the reliability of the method due to two potential modifications of the yield from off-shell effects and initial and final state interactions in the TH $2 \rightarrow 3$ reaction. In the TH reaction, shown schematically in Fig. 1, particle x in the binary subprocess $x + A \rightarrow b + B$ is virtual (off-energy-shell). In the standard analysis, the virtual nature of x is
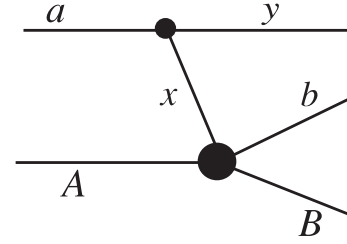

FIG. 1: The pole diagram describing the TH reaction $a + A \rightarrow y + b + B$.

neglected and the plane wave approximation is used [3, 5]. Here we address, for the first time, the reliability of this assumption. We also consider scattering between particles a and A in the initial channel of the TH reaction and the dominance of the QF mechanism. Note that the TH reaction is a many-body process (at least four-body) and its strict analysis requires many-body techniques. However some important features of the TH method can be addressed in a simple model.

First the notation for the subsequent discussion is introduced. $E_{ij} (= k_{ij}^2 / 2\,\mu_{ij})$ and k_{ij} are the relative energy and momentum of the real particles i and j, $\mu_{ij} = m_i\,m_j / (m_i + m_j)$, where m_i is the mass of particle i, and $\mathbf{p}_{xj}$ is the relative momentum of the virtual particle x and particle j. The internal particle x of the diagram shown in Fig. 1 is virtual, i.e. $E_x \neq p_x^2 / 2\,m_x$. From energy-momentum conservation laws in the three-ray vertex $a \rightarrow x + y$ and the four-ray vertex $x + A \rightarrow b + B$, we get $\sigma_x = p_{xy}^2 / (2\,\mu_{xy}) + \varepsilon_{xy}^a = p_{xA}^2 / (2\,\mu_{xA}) - E_{xA}$, where $\varepsilon_{xy}^a = m_x + m_y - m_a$ is the binding energy for the virtual decay $a \rightarrow x + y$. Thus in a TH reaction $p_{xA}^2 / (2\,\mu_{xA}) > E_{xA} = k_{xA}^2 / (2\,\mu_{xA})$ holds. The reaction amplitude corresponding to the diagram is given by (all particles are assumed, for simplicity, to be spinless) $M_p = M_{2 \rightarrow 2}^{(HOF)}\,\varphi_{xy}(p_{xy})$. Here $M_{2 \rightarrow 2}^{(HOF)}(\sigma_x, E_{xA}, z)$ is the half-off-energy-shell (HOF) reaction amplitude for the direct binary subprocess $x + A \rightarrow b + B$ which depends on the additional variable σ_x due to the virtual nature of particle x. Also $z = \hat{\mathbf{p}}_{xA} \cdot \hat{\mathbf{k}}_{bB}$, with $\hat{\mathbf{k}} = \mathbf{k}/k$. The Fourier transform of the s-wave bound-state wave function for $a = (x\,y)$ can

be written as $\varphi_{xy}(p_{xy}) = -W_{xy}(p_{xy})/(\sigma_x)^{1-\eta_{xy}}$, where η_{xy} is the Coulomb parameter of the bound state (xy) and W_{xy} is the amplitude for the virtual decay $a \rightarrow x+y$, which is regular at the singular point $\sigma_x = 0$. This singularity is a branch point for the charged particles x and y. For $a = d = (pn)$, $\varphi_{xy}(p_{xy})$ has a pole at $\sigma_x = 0$, corresponding to the real (on-energy-shell (ON)) particle x. Hence the name pole diagram for Fig. 1 [6]. The modulus of the amplitude $|M_p|$ of the pole diagram has a maximum at $p_{xy} = 0$ called the QF peak and the condition corresponding to $p_{xy} = 0$ is called QF kinematics. The QF peak is a trace of the pole located at $p_{xy}^2 < 0$. The QF kinematics are ideal for the TH method since small p_{xy} corresponds to large separation distance between particles x and y, thereby allowing particle y to be treated as a spectator.

By dropping all the terms containing particle y and its interaction with other nuclei, the half-off-shell post-form amplitude of the direct binary reaction, $A(x,b)B$, can be extracted from the exact amplitude for the TH reaction, $A(a,y\,b)B$. The result is given by

$$M_{2\rightarrow2}^{(HOF)}(\sigma_x, E_{xA}, z) \sim <\psi_{bB}^{(-)}\varphi_b\varphi_B|\hat{O}|\varphi_x\varphi_A\mathrm{p}_{xA}> . \tag{1}$$

Here, $\psi_{bB}^{(-)}$ is the distorted wave in the final state of the binary process, φ_i is the bound state wave function of nucleus i, $\hat{O} = \Delta V_{bB}[1 + G_{xA}^{(+)}\Delta V_{xA}]$ is the transition operator, $\Delta V_{ij} = V_{ij} - U_{ij}$, V_{ij} (U_{ij}) is the interaction (optical) potential between nuclei i and j and $G_{xA}^{(+)}$ is the Green's function of the system $x + A$ (or $b + B$). The half-off-shell amplitude contains the off-shell plane wave $|\mathrm{p}_{xA}>$ which describes the relative motion of the virtual particle x and A in the initial channel of the binary reaction rather than the distorted wave describing the initial state in the on-shell amplitude. Hence the half-off-shell amplitude does not contain a Coulomb barrier factor. First we estimate the contribution from the transition operator $[\Delta V_{bB}\,G_{xA}^{(+)}\,\Delta V_{xA}]$ to $M_{2\rightarrow2}^{(HOF)}$. Consider two reactions previously analyzed using the TH method, $^6\mathrm{Li}(d,\alpha)^4\mathrm{He}$ $(Q_{2\rightarrow2} = 21.64$ MeV) [3] and $^7\mathrm{Li}(p,\alpha)^4\mathrm{He}$ $(Q_{2\rightarrow2} = 16.56$ MeV) [7]. The first reaction was treated as a deuteron transfer and the second as a triton transfer. Nuclei A and b are bound states $A = (B\,t)$ and $b = (x\,t)$ with constituent particles x, t, B, where t is the transferred particle. In this simple model the half-off-shell amplitude takes the form (all particles are assumed to be spinless)

$$M_{2\rightarrow2}^{(HOF)}(\sigma_x, E_{xA}; z) \sim <\psi_{bB}^{(-)}\varphi_{xt}|\hat{O}|\varphi_{Bt}\mathrm{p}_{xA}> . \tag{2}$$

The bound state wave functions have been approximated by their tails, $\Delta V_{bB} \approx V_{Bt}$ and the zero-range approximation (ZRA) has been used for $V_{Bt}\varphi_{Bt}$. The ZRA is good enough to determine the energy dependence of the cross section. At $E_{xA} < 1$ MeV, only the Coulomb interaction needs to be included in the transition operator ΔV_{xA} and in the Green's function. Also at low E_{xA} the dominant contribution comes only from the s-wave in the
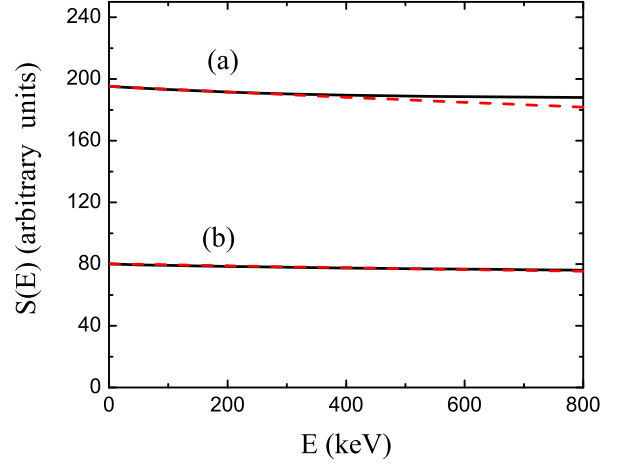

FIG. 2: (Color online). Energy dependence $(E \equiv E_{xA})$ of the half-off-shell (red dashed line) and on-shell (black solid line) astrophysical factors for (a) the $^7\mathrm{Li}(p,\alpha)^4\mathrm{He}$ reaction $(a = d, x = p, y = n)$; (b) the $^6\mathrm{Li}(d,\alpha)^4\mathrm{He}$ reaction $(a = {}^6\mathrm{Li}, x = d, y = \alpha)$.

channel $x + A$ due to the large $Q_{2\rightarrow2}$ value for both reactions. Consequently the z dependence of $M_{2\rightarrow2}^{(HOF)}$ can be neglected. The contribution from the transition operator $[\Delta V_{bB}G_{xA}^{(+)}\Delta V_{xA}]$ is negligible for both reactions as is the case for the on-shell processes. It constitutes a background for the DWBA with the transition operator given by ΔV_{bB}.

The on-shell amplitude to be compared with the half-off-shell one is

$$M_{2\rightarrow2}^{(ON)}(E_{xA}) \sim <\psi_{bB}^{(-)}\varphi_{xt}|\Delta V_{bB}|\varphi_{Bt}\psi_{xA}^{(+)}> . \tag{3}$$

Here, $\psi_{xA}^{(+)}$, the distorted wave describing the scattering of real particles x and A in the initial channel, can be approximated by the pure Coulomb scattering wave function at low energies. Consequently, $\psi_{xA}^{(+)} = N_{xA}\,exp(i\,\mathbf{k}_{xA}\cdot\mathbf{r}_{xA})\,_1F_1$, where N_{xA} is the Gamow normalization factor and $_1F_1$ is the hypergeometric function, which has a very weak energy dependence at small k_{xA}. The only difference in the energy dependence between the half-off-shell and on-shell astrophysical factors comes from the the use of the half-off-shell plane wave in the initial state in Eq. (2) and the on-shell distorted wave in Eq. (3). From Fig. 2 it is clear that the results, which were calculated for QF kinematics, clearly justify the TH method. The energy dependence of the half-off-shell and on-shell astrophysical factors at low energies are practically identical. Since only the energy dependence is of interest, the half-off-shell result in Fig. 2 has been normalized to the on-shell one at an energy $E_{xA} = 1$ keV for ease of comparison.

Another way to determine the effect of the virtual nature of the transferred particle is to vary the parameter σ_x. In Fig. 3 (a) the energy dependence of the half-off-shell astrophysical factor for the $^6\mathrm{Li}(d,\alpha)^4\mathrm{He}$ reaction
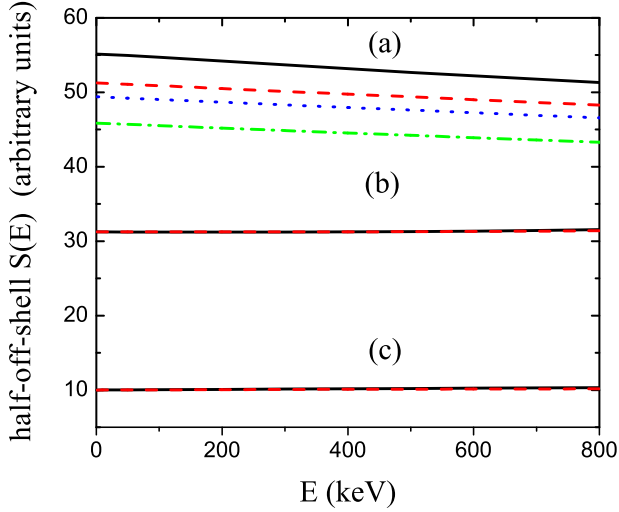
FIG. 3: (Color online). (a) Energy dependence ($E \equiv E_{d^6\text{Li}}$) of the half-off-shell astrophysical factor for the $^6\text{Li}(d,\alpha)^4\text{He}$ reaction as a function of σ_d: $\sigma_d = 0$- black solid line (on-shell kinematics); $\sigma_d = \varepsilon_{d\alpha}^{^6\text{Li}}$- red dashed line (QF kinematics); $\sigma_d = 1.5\,\varepsilon_{d\alpha}^{^6\text{Li}}$ - blue dotted line; $\sigma_d = 2.5\,\varepsilon_{d\alpha}^{^6\text{Li}}$ - green dashed-dotted line. (b) Comparison of the energy dependence of the half-off-shell astrophysical factor for the $^6\text{Li}(d,\alpha)^4\text{He}$ reaction determined from the $^6\text{Li}(^6\text{Li},\alpha\,\alpha)^4\text{He}$ TH reaction with (black solid line) and without (red dashed line) inclusion of the initial $^6\text{Li} - {}^6\text{Li}$ Coulomb interaction. (c) The same as in (a) but with the final-state interaction described by the Ali-Bodmer potential: $\sigma_d = 0$- black solid line (on-shell kinematics); $\sigma_d = \varepsilon_{d\alpha}^{^6\text{Li}}$- red dashed line (QF kinematics). In both cases (b) and (c) the red dashed line is normalized to the black solid line at $E = 1$ keV.

($x = d; A = {}^6\text{Li}; b = B = \alpha$) is plotted for different values of σ_d. Note that $\sigma_d = 0$ corresponds to the on-shell deuteron. The deuteron becomes farther from the energy-shell as σ_d grows. It is clear from Fig. 3 that the size of the S factor changes but the energy dependence does not change. This justifies the procedure of disregarding the virtual nature of the entry particle in the analysis of the binary process $A(x,b)B$ [3, 5]. If the amplitude of the TH process $A(a,y\,b)B$ is approximated by the pole amplitude M_p, as is done in the TH method, then the amplitude of the binary subprocess can be trivially singled out. This approximation ignores the interaction between particles a and A in the entry channel. Inclusion of the Coulomb interaction in the $a - A$ channel is a way to gauge the validity of this approximation. Note that including the Coulomb-nuclear interaction in the exit channel does not change the final result. The TH reaction amplitude with Coulomb scattering in the entrance channel is given by

$$M^{(TH)} \sim\; <\psi_{yF}^{(-)}\psi_{bB}^{(-)}\varphi_{xt}|\Delta V_{bB}|\varphi_{Bt}\,\varphi_{xy}\,\psi_{aA}^{(C)(+)}>, \quad (4)$$

where $\psi_{yF}^{(-)}$ is the distorted wave describing the scattering of y in the center-of-mass of the system $F = b + B$. As an example, consider the TH reaction $^7\text{Li}(d,n\,\alpha)^4\text{He}$, where $a = d = (pn)$, $A = {}^7\text{Li}$. The bound state wave function

of d is approximated by its tail. The Coulomb interaction between $d-{}^7\text{Li}$ modifies the behavior of the amplitude of the TH reaction. For simplicity, the final state scattering wave functions are replaced by plane waves. To single out the amplitude of the binary process $^7\text{Li}(p,\alpha)^4\text{He}$ in QF kinematics, the $M^{(TH)}$ is divided by the factor

$$R =\; <\varphi_{xy}(\mathbf{k}_{yF} - (m_y/m_a)\,\mathbf{p}_{aA})|\psi_{\mathbf{k}_{aA}}^{(C)(+)}(\mathbf{p}_{aA})>, \quad (5)$$

where integration is performed over $\mathbf{p}_{aA}$ and $\psi_{\mathbf{k}_{aA}}^{(C)(+)}(\mathbf{p}_{aA})$ is the Fourier component of the $a - A$ Coulomb scattering wave function. This factor determines the behavior of $M^{(TH)}$ near the singularity $\sigma_x = 0$. Note that for the results presented in Fig. 2 the half-off-shell astrophysical factor has been calculated at a fixed $p_{xy} = 0$ corresponding to QF kinematics of the TH $2 \rightarrow 3$ reaction. E_{xA} becomes a unique function of E_{aA}. In practice TH experiments are carried out at a fixed energy, E_{aA}, while p_{xy} is allowed to vary around $p_{xy} = 0$. For example, to cover the energy region $0 < E_{p^7\text{Li}} \leq 800$ keV for the $^7\text{Li}(d, n\alpha)^4\text{He}$ reaction at energy $E_{d^7\text{Li}} = 4.20$ MeV, it is sufficient to vary the factor σ_p by 7% around the QF value $\sigma_p = 2.224$ MeV.

To test the importance of the Coulomb interaction in the initial state, the energy dependence of the half-off-shell astrophysical factor for the binary $^6\text{Li}(d,\alpha)^4\text{He}$ reaction, determined from the TH reaction $^6\text{Li}(^6\text{Li},\alpha\,\alpha)^4\text{He}$ at $E_{^6\text{Li}^6\text{Li}} = 3.14$ MeV, the TH reaction amplitude (Eq. 4) was calculated with and without the initial Coulomb interaction term. The results, as seen in Fig. 3 (b), show that the energy dependence of both astrophysical factors is nearly identical. Thus including the Coulomb interaction in the initial state does not affect the energy dependence of the astrophysical factor determined from the TH reaction. Similar calculations done for $S(E_{p^7\text{Li}})$ for the binary $^7\text{Li}(p,\alpha)^4\text{He}$ reaction, determined from the TH reaction $^7\text{Li}(d, n\,\alpha)^4\text{He}$ at $E_{d^7\text{Li}} = 4.03$ MeV, show the same result. As also seen from Fig. 3 (c), the final state interaction (here described by the Ali-Bodmer $\alpha - \alpha$ potential) does not affect the energy behavior of the on-shell and half-off-shell astrophysical factors for the binary process $^6\text{Li}(d,\alpha)^4\text{He}$.

Understanding TH reactions proceeding through a resonant binary subprocess is simpler than for the direct case. The TH pole mechanism for the direct process near the QF peak is dominated by the singularity (pole for $a = d$ or branch point for charged particles x and y) in the bound state wave function $\varphi_{xy}(p_{xy})$. In a resonant reaction, the mechanism depends on two singularities, one that occurs in the bound state wave function in the p_{xy} plane and the other that is due to the resonant pole of the amplitude of the binary subprocess $A(x,b)B$ in the E_{xA} plane. The TH $2 \rightarrow 3$ reaction proceeding through the resonance in the subsystem $x + A$ is very similar to stripping (d,p), (d,n) or α-transfer $(^6\text{Li}, d)$ reactions populating resonant states [8]. The difference is that in the TH process we are interested in the decay of the intermediate resonances into the channel $b + B$ which differs from the entry channel $x + A$. The half-off-shell
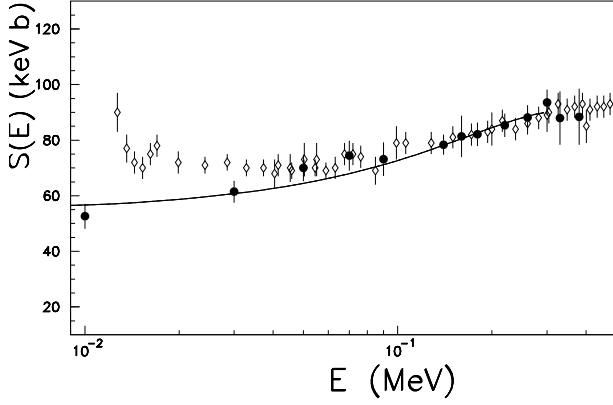
FIG. 4: $S(E)$ for the ^{7}Li$(p,\alpha)^4$He reaction. The solid curve is the polynomial fit to the data (full circles) obtained from the TH ^{7}Li$(d,\alpha\,n)^4$He [7] reaction. Open cirles are the data from the direct measurement which include electron screening effects [9].

s-wave resonant amplitude is given by

$$M^{(HOF)(R)}(k_{bB}, p_{xA}; E) = -2\,\pi\,\sqrt{\frac{1}{\mu_{bB}k_{bB}}}\,e^{i\delta_{f0}(k_{bB})}$$
$$\times\frac{\sqrt{\Gamma_{bB}(E_{bB})}\,w_0(p_{xA}, k_{xA(R)})}{E_{xA} - E_{xA}^{(R)}}. \quad (6)$$

Here, $\Gamma_{bB}(E_{bB})$ is the partial width of the resonance F^* in the channel $b + B$, $k_{xA(R)} = \sqrt{2\,\mu_{xA}\,E_{xA}^{(R)}}$, $E_{xA}^{(R)}$ is the resonance energy and δ_{f0} is the non-resonant (potential) scattering phase shift of particles b and B in the final state. The off-shell s-wave form factor for the vertex $x + A \rightarrow F^*$ is given by $w_0(p_{xA}, k_{xA(R)}) = [E_{xA}^{(R)} - p_{xA}^2/(2\,\mu_{xA})]\,\psi_{n0}^{(R)}(p_{xA})$, where $\psi_{n0}^{(R)}(p_{xA}) = <j_0(p_{xA}\,r)|\psi_{n0}^{(R)}(r)>$ is the Fourier transform of the Gamow radial wave function $\psi_{n0}^{(R)}(r)$, $j_0(p_{xA}\,r)$ is the s-wave spherical Bessel function and n is the principal quantum number. Due to factorization of the off-shell form factor in the half-off-shell resonant amplitude, the ratio of the half-off-shell to on-shell astrophysical factors is $S^{(HOF)(R)}/S^{(ON)(R)} = k_{xA}\,w_0(p_{xA}, k_{xA(R)})/[(e^{-2\pi\,\eta_{xA}})\Gamma_{xA}(E_{xA})]$, where η_{xA} is the Coulomb parameter. Taking into account that $\Gamma_{xA}(E_{xA}) \sim k_{xA}\,P_0(E_{xA})$, where $P_0(E_{xA})$ is the barrier penetrability, we conclude that the energy dependence of

the half-off-shell and on-shell astrophysical factors at low energies is the same. Note that if the Coulomb interaction in the initial state is included, the half-off-shell resonant reaction amplitude should be divided by the same factor R as for direct processes.

A simple model has been used to compare the energy behavior of the direct and resonant half-off-shell and on-shell astrophysical factors. The intention was not to reproduce experimental data, which requires a more sophisticated approach, but to demonstrate that the energy dependence of the half-off-shell and on-shell astro-
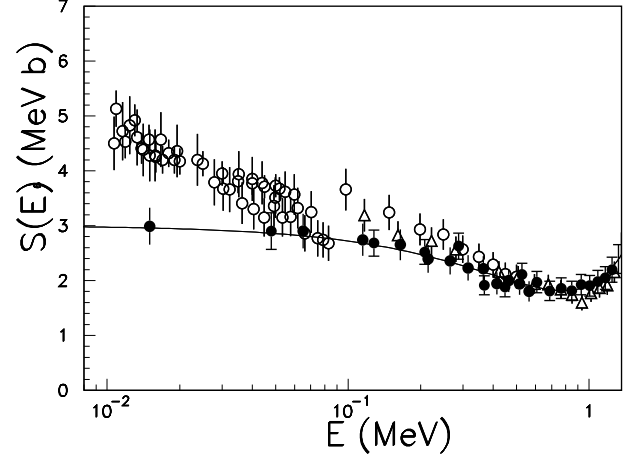

FIG. 5: $S(E)$ for ^{6}Li$(p,\alpha)^3$He $S(E_{p^6\mathrm{Li}})$ from the TH reaction ^{6}Li$(d,\alpha\,n)^3$He (solid circles) [10] compared to direct data (open triangles [11] and open circles [9]). The line shows the result of a second order polynomial fit to the TH reaction data [10].

physical factors are nearly identical when analyzed in the same model. Validating this makes it clear why the TH method is such a powerful indirect technique for nuclear astrophysics. The power of the TH method is seen in Figs 4 and 5. In both cases the energy dependence of the astrophysical factor determined from the TH reaction nicely reproduces the energy dependence of the astrophysical factor obtained from direct measurements at higher energies. At lower energies the TH method provides the astrophysical factor between bare nuclei while the direct data are distorted by electron screening.

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